4) Differential graded categories

$*$ Recall that a differential graded category (dg-category for short) is a category enriched over Cpl_R for some $\wedge$ ring R.

commutative.

- $Ob(Cpl_R)$: (C_*, ∂) with $\begin{cases} C_n \in Mod_R & (n \in \mathbb{Z}) \\ \partial_n : C_n \to C_{n-1} \\ \text{with } \partial_n \circ \partial_{n+1} = 0 \end{cases}$

chain complex

boundary operator

- $$Cpl_R((C_*, \partial), (D_*, \partial)) = \left\{ f_n : C_n \to D_n \;\middle|\; \begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\equiv"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial_n"] & D_{n-1} \end{tikzcd} \right\}$$

- $$(C_* \otimes D_*)_n := \bigoplus_{n = n' + n''} C_{n'} \otimes D_{n''}$$

and $$\partial(\underset{C_{n'}}{\underset{\cap}{c}} \otimes \underset{D_{n''}}{\underset{\cap}{d}}) := \partial(c) \otimes d + (-1)^{n'} c \otimes \partial(d)$$

- $$H_n(C_*) = \frac{Ker(\partial_n : C_n \to C_{n-1})}{Im(\partial_{n+1} : C_{n+1} \to C_n)}$$ homology groups

- $f : C_* \to D_*$ is a quasi-isomorphism if $\forall n \in \mathbb{Z}$,
  $H_n(f) : H_n(C_*) \xrightarrow{\sim} H_n(D_*)$.

* dg-categories are very common in homological algebra and algebraic geometry, in the same way that simplicial and topological categories are in algebraic topology.

* We write $dgCat_R$ for the category of R-linear dg-categories.

Ex 24:

* Let $\mathcal{A}$ be an R-linear additive category, i.e a category enriched over Mod_R and which admits all finite products. Then those finite products are automatically coproducts (see Ex. sheet 9) Hence $\mathcal{A}$ has in particular a zero object (= initial + final). Using this, one can define the category $Ch(\mathcal{A})$ of chain complexes in $\mathcal{A}$

as above (so that $Cpl_R = Cpl(Mod_R)$).

Then $Cpl(\mathcal{A})$ admits a natural structure of R-linear dg-category, with

$$\overset{\text{mapping complex}}{\underline{\mathrm{Hom}}}_{Cpl(\mathcal{A})}(C_*, D_*)_n = \prod_{i \in \mathbb{Z}} \overset{\substack{\mathrm{Mod}_R \\ \cup}}{\underline{\mathrm{Hom}}_{\mathcal{A}}}(C_i, D_{i+n})$$

and boundary operator:

$$\partial\{f_i : C_i \longrightarrow D_{i+n}\}_i = \{\partial \circ f_j - (-1)^j f_{j-1} \circ \partial\}_j$$

(In particular, Cpl_R is canonically enriched over itself).

* A dg-R-algebra A_* is a (non-necc. commutative) graded R-algebra + a differential $\partial : A_* \longrightarrow A_{*-1}$ satisfying

· $\partial^2 = 0$

· $\partial(xy) = \partial(x)\, y + (-1)^{|x|} x\, \partial y$ (Leibniz rule)

Then
$$\left\{ \begin{matrix} \text{dg-}R\text{-categories} \\ \text{with one object } X \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{dg-}R \\ \text{algebras} \end{matrix} \right\}$$
$$C \longmapsto \operatorname{End}_C(X)_*$$

* As with Cat_Δ, $dgCat_R$ is naturally connected with other types of enriched categories via lax-monoidal functors.

We have:

$$\cdot \quad Cpl_R(\underset{(\mathbb{1}_{Cpl_R})}{\underset{\parallel}{R[0]}}, C_*) = \overset{\text{0-cycles}}{\overset{\downarrow}{Z_0}}(C_*)$$

$$\cdot \qquad = \{ x \in C_0 \mid \partial(x) = 0 \}$$

$$\Rightarrow \quad u : dgCat_R \longrightarrow Cat_R$$

underlying category functor

- The 0-th homology functor:

$$H_0 : \mathrm{Cpl}_R \longrightarrow \mathrm{Mod}_R$$

$$C_* \longmapsto H_0(C_*) := \frac{\overset{Z_0(C_*)}{\overset{\shortparallel}{\mathrm{Ker}(d_0 : C_0 \to C_{-1})}}}{\underset{B_0(C_*)}{\underset{\shortparallel}{\mathrm{Im}(d_1 : C_1 \to C_0)}}}$$

H_0 has a natural lax-monoidal structure

$\rightsquigarrow$ homotopy category functor

$$\mathcal{R} = (H_0)_* : \mathrm{dgCat}_R \longrightarrow \mathrm{Cat}_R$$

Applied to $\mathrm{Cpl}(\mathcal{A})$, this encodes nicely the notion of chain homotopy of morphisms of complexes which plays a key role in homological algebra.

- We want to construct interesting ∞-categories using dg-categories.

Here is a procedure to do it:

def 25: Let C be in $dgCat_R$. The **differential graded nerve** $N^{dg}(C)$ is the simplicial set defined as follows:

- $N^{dg}(C)_n$ is the collection of tuples $((X_i)_{0 \leq i \leq n}, \{f_I\})$ with $X_i \in Ob(C)$

and, for every $I = \{i_0 > i_1 \cdots > i_k\} \subseteq [n]^{op}$ with $|I| \geq 2$, we have $f_I \in Hom_C(X_{i_k}, X_{i_0})_{k-1}$

which satisfy

$$\partial f_I = \sum_{a=1}^{k-1} (-1)^a \left(f_{\{i_0 > i_1 \cdots > i_a\}} \circ f_{\{i_a > \cdots i_k\}} - f_{I \setminus \{i_a\}} \right)$$

- The simplicial structure is given by $\alpha : [n] \longrightarrow [m]$,

$$\alpha^*\left(\{x_i\}, \{g_I\} \right) = \left(\{x_{\alpha(i)}\}, \{g_J\} \right)$$

with
$$g_J = \begin{cases} g_{\alpha(J)} & \text{if } \alpha_{|J} \text{ is injective} \\ \mathrm{id}_{x_i} & \text{if } J = \{j_0 > j_1\} \text{ with } \alpha(j_0) = \alpha(j_1) = i \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Ex 26: Rather than getting lost in a mess of indices, let's look concretely at the n-simplices for $n \leq 2$:

- $N^{dg}(C)_0 = \mathrm{Ob}(C)$
- $N^{dg}(C)_1 = \mathrm{Mor}(\upsilon C)$ $\left(= 0\text{-cycles in } \mathrm{Hom}_C(x,y) \right)$
- $N^{dg}(C)_2 = \{ (x_0, x_1, x_2),\ g_{01} \in Z_0(x_0,x_1),$
  $g_{02} \in Z_0(x_0,x_2),\ g_{12} \in Z_0(x_1,x_2)$

and $g_{012} \in \mathrm{Hom}_C(X_0, X_2)_1$ such

that $\partial g_{012} = g_{02} - g_{12} \circ g_{01} \}$

- Like in the homotopy coherent nerve, we can think of this datum as

$$\begin{tikzcd}
 & X_1 \arrow[dr, "g_{12}"] \arrow[d, phantom, ""{name=U}] & \\
X_0 \arrow[ur, "g_{01}"] \arrow[rr, "g_{02}"', ""{name=D}] & {} & X_2
\arrow[Rightarrow, from=D, to=U, "g_{012}"']
\end{tikzcd}$$

with g_{012} witness of an equality of homology classes : $[g_{02}] = [g_{12} \circ g_{01}]$ in $H_0(\mathrm{Hom}_C(X_0, X_2))$.

Thm 27: Let C be a dg-category. Then $N^{dg}(C)$ is an ∞-category. $\square$

The proof is a straightforward

but lengthy computation.

- Thm 27, combined with standard constructions from homological algebra, produces many interesting ∞-categories.

  ⚠ Unlike in the simplicial case, not every ∞-category arises this way, even up to categorical equivalence! One should think of the essential image as the "R-linear ∞-categories".

- For instance, let $\mathcal{A}$ be an abelian category with enough injectives (e.g $\mathcal{A} = \mathrm{Mod}_R$, or $\mathcal{A} = \mathrm{QCoh}_X$ for a scheme X) Let $\mathrm{Cpl}^+(\mathcal{A}_{inj})$ be the full subcategory of $\mathrm{Cpl}(\mathcal{A})$ formed by the C_*

with : - C_n injective for all $n \in \mathbb{Z}$

- $C_n = 0$ for $n >> 0$

(bounded above (homologically) complexes of injectives)

Then $Cpl^+(\mathcal{A}_{inj})$ "inherits" a dg-enrichment from $Cpl(\mathcal{A})$.

def 28: The (bounded-above) derived ∞-category $\mathcal{D}^+(\mathcal{A})$ is defined as

$$N^{dg}(Cpl^+(\mathcal{A}_{inj})).$$

(there are variants for bounded, bounded below and unbounded complexes as well).

* Essentially by construct°, the homotopy category of $\mathcal{D}^+(\mathcal{A})$ is

$$R\ \mathcal{D}^+(\mathcal{A}) = K^+(\mathcal{A}_{inj})$$

$$= D^+(\mathcal{A}) \leftarrow \text{usual derived category}$$

Natural questions:

+ Is the triangulated structure on $D^+(\mathcal{A})$ somehow encoded in $\mathcal{D}^+(\mathcal{A})$?

  Yes! $\mathcal{D}^+(\mathcal{A})$ is a stable ∞-category and stable ∞-categories have a canonical triangulated structure on their homotopy categories.

+ What is the advantage of working with $\mathcal{D}^+(\mathcal{A})$ rather than $D^+(\mathcal{A})$?

  Homotopy coherence! Makes it possible to talk about $\left\{ \begin{array}{l} \text{sheaves with values in } \mathcal{D}^+(\mathcal{A}) \\ \text{higher algebraic structures} \ \text{——} \end{array} \right.$

$\rightsquigarrow$ derived algebraic geometry (in char. 0)

5) Dold-Kan correspondence *

- There is a fundamental relationship between chain complexes and simplicial abelian groups / R-modules.

def 29: Let $A_\bullet \in s\mathrm{Mod}_R$ be a simplicial R-module. Its Moore complex is $C_*(A_\cdot) \in \mathrm{Cpl}_R$ defined by

$$C_n(A) = \begin{cases} A_n, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

and $\partial : A_n \longrightarrow A_{n-1}$

$$\sigma \longmapsto \sum_{i=0}^{n} (-1)^i d_i(\sigma)$$

$$\begin{tikzcd}
d_0 \arrow[d] & A_0 \arrow[r] & A_1 \arrow[l, shift right=3, "+"'] \arrow[l, shift left=3, "-"] \arrow[r, shift left=1] \arrow[r, shift right=1] & A_2 \arrow[l, shift right=5, "+"'] \arrow[l, "-"] \arrow[l, shift left=5, "+"] & \circ \circ \circ \\
d_n & & & &
\end{tikzcd}$$

(Exercise : show $\partial^2 = 0$)

def 30 : The free R-module functor is $s\mathrm{Set} \xrightarrow{R[-]} s\mathrm{Mod}_R$

$$(S : \Delta^{op} \to \mathrm{Set}) \longmapsto R[S] : \Delta^{op} \to \mathrm{Set} \xrightarrow{R^{(-)}} \mathrm{Mod}_R$$

(NB: this is strong monoidal : $R[S \times T] \cong R[S] \otimes R[T]$)

def 31 : Let $S \in s\mathrm{Set}$. The chain complex of X. (or associated to X.) is

$$C_*(S, R) := C_*(R[S]).$$

The homology R-modules of X are

$$H_n(S, R) := H_n\, C_*(S, R).$$

Ex 32 :

- Let $X \in \mathrm{Top}$. Then $C_*(\mathrm{Sing}(X), R)$ is the usual singular chain complex $C_*^{\mathrm{sing}}(X, R)$

and $H_*(X, R)$ is singular homology.

- Let $S \in sSet$. We have a unit map

$$S \longrightarrow Sing|S|$$

which induces a quasi-isomorphism.

$$C_*(S, R) \longrightarrow C_*^{sing}(|S|, R)$$

In other words, $C_*(S, R)$ is a simplicial way to capture the singular homology of $|S|$.

def 33: Let $A_\cdot \in sMod_R$. The degenerate sub complex $D_*(A) \subseteq C_*(A)$ is $D_n(A) = \langle s_i : A_{n-1} \to A_n \mid 0 \leq i \leq n-1 \rangle$

(Exercise: it is indeed a subcomplex, preserved by ∂)

The normalised Moore complex $N_*(A)$ is

$$N_*(A) := \frac{C_*(A)}{D_*(A)}$$

Prop 34: The quotient map $C_*(A) \to N_*(A)$ is a quasi-isomorphism. $\square$

Ex 35: We write $N_*(-,R)$ for the composition

$$N_*(-,R) : s\,Set \xrightarrow{R[-]} s\,Mod_R \xrightarrow{N_*} Cpl_R$$

By construction, $N_*(\Delta^\bullet, R)$ is then a cosimplicial object in Cpl_R.

It is in fact relatively simple

$$N_*(\Delta^n, R)_i \simeq R\left[\begin{array}{l}\text{non-degenerate } i\text{-th simplices} \\ \text{of } \Delta^n\end{array}\right] \simeq R^{\binom{n}{i}}$$

and

$$\begin{aligned} \partial : N_i(\Delta^n) &\longrightarrow N_{i-1}(\Delta^n) \\ \sigma &\longmapsto \sum_{i=0}^{n} (-1)^i \begin{cases} d_i(\sigma) & \text{if non-deg.} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\left\{\begin{array}{l} N_*(\Delta^0) : \cdots 0 \to 0 \to \overset{\overset{0}{\downarrow}}{R} \to 0 \to \cdots \\[1em] N_*(\Delta^1) : \cdots 0 \to R \oplus R \xrightarrow{\binom{1}{-1}} R \to \cdots \end{array}\right.$$

We use this cosimplicial object to construct:

def 36 : The Eilenberg-Maclane Functor

$K : Cpl_R \longrightarrow sMod_R$ is defined by

$$K(C_*)_n := Cpl_R(N_*(\Delta^n, R), C_*)$$

Rmk 37 : * As with every simplicial group , the underlying simplicial set of $K(C_*)$ is always a Kan complex. (See exercise 3.8)

* If $C_* = A[n]$, $n \geq 1$ with $A \in Ab$, $|K(A[n])|$ is an Eilenberg-Maclane space in the sense of algebraic topology:

$$\pi_i(|K(A[n])|) \simeq \begin{cases} * & , i=0 \\ A & , i=n \\ 0 & , \text{otherwise.} \end{cases}$$

Prop 38: K is the right adjoint of N_*.

Thm 39 (Dold-Kan correspondence)

$N_* \dashv K$ restricts to an adjoint equivalence

$$N_* : s\mathrm{Mod}_R \underset{\sim}{\rightleftarrows} \mathrm{Cpl}_{R,\geq 0} : K \qquad \square$$

This is surprisingly (to me!) difficult to prove for such an "abstract" result.

(see [Kerodon, 2.5.6])

- The interaction with the symmetric monoidal structures on both sides is also a surprisingly complicated story, which we don't have time to discuss. We just extract the result we need:

Thm 40 (Some monoidal aspects of Dold-Kan)

The functors N_* and K both admit natural lax-monoidal structures. (called the Eilenberg-Zilber and Alexander-Whitney map). $\square$

Using those, we can finally move between simplicial and dg-categories: we get an adjunction

$$R[-] := (N_* \circ R[-])_* : \mathrm{Cat}_\Delta \rightleftarrows \mathrm{dgCat}_R : (\mathrm{forget} \circ K)_* : u$$

(⚠ This is not at all an equivalence!)

Note that for any $\mathcal{C} \in \mathrm{dgCat}_R$, the simplicial category $u\,\mathcal{C}$ is locally Kan.

Combined with the homotopy cohent nerve, this gives a functor

$$N_{\Delta} U \ : \ dgCat_{R} \longrightarrow Cat_{\infty} .$$

Thm 41: For any $C \in dgCat_R$, there is a functor

$$N_{\Delta}(UC) \longrightarrow N^{dg}(C)$$

which is a trivial fibration, and thus an equivalence of ∞-categories. □

This shows that the two "natural" ways to make ∞-categories out of a dg-category coincide.

6) Model categories

Model categories are yet another way to present homotopical / ∞-categorical information with 1-categories.

def 42: (Quillen) A model category is a category M together with 3 collections of morphisms:

- the weak equivalences W
- the cofibrations Cof
- the fibrations Fib

such that the following axioms hold:

(not essential, could ask for finite (co)limits)

- M is complete and cocomplete.
- W satisfies the 2-out-of-3 property; if f, g are composable and 2 out of 3

among $f, g, f \circ g$ are in W, so is the third.

- $\left(\mathrm{Cof}, \overset{\substack{\text{trivial fibrations} \\ \downarrow}}{\mathrm{Fib} \cap W}\right)$ and $\left(\overset{\substack{\text{trivial cofibrations} \\ \swarrow}}{\mathrm{Cof} \cap W}, \mathrm{Fib}\right)$

  are weak factorisation systems:

  every morphism $f : X \to Y$ in M factors (non-uniquely) as

$$\left\{\begin{array}{l} f : X \underset{\mathrm{Cof}}{\hookrightarrow} P \underset{\mathrm{Fib} \cap W}{\overset{\sim}{\twoheadrightarrow}} Y \\[2em] f : X \underset{\mathrm{Cof} \cap W}{\overset{\sim}{\hookrightarrow}} C \underset{\mathrm{Fib}}{\twoheadrightarrow} Y . \end{array}\right.$$

and we have lifting properties:

$$\left\{\begin{array}{l} \mathrm{Cof} = {}^{\boxslash}(\mathrm{Fib} \cap W) \\[2em] \mathrm{Cof} \cap W = {}^{\boxslash}\mathrm{Fib} \end{array}\right. \iff \begin{array}{l} \mathrm{Fib} \cap W = \mathrm{Cof}^{\boxslash} \\[2em] \mathrm{Fib} = (\mathrm{Cof} \cap W)^{\boxslash} . \end{array}$$

$\square$

Remarks 43:

* The definition is self-dual:

(M, W, Cof, Fib) is a model category

$\times \quad \Updownarrow$

$(M^{op}, W^{op}, Fib^{op}, Cof^{op})$ ———————— .

$*$ X is cofibrant (resp. fibrant) if the unique map $\emptyset \longrightarrow X$ (resp. $X \longrightarrow *$) is a cofibration (resp. a fibration).

$*$ Using the factorisations, we see that any object $X \in M$ admits:

- $X_{cof} \overset{\sim}{\twoheadrightarrow} X$ with X_{cof} cofibrant (a cofibrant replacement | resolution of X)
- $X \overset{\sim}{\hookrightarrow} X_{Fib}$ with X_{Fib} fibrant. (a fibrant replacement | resolution of X).

By combining the two, we get that any object X is weak equivalent to a bifibrant (= fibrant + cofibrant) object:

$$\begin{tikzcd} X & X_{cof} \arrow[l, two heads, "\sim"'] \arrow[r, hook, "\sim"] & (X_{cof})_{fib} \end{tikzcd}$$

$$\left(\text{or} \quad \begin{tikzcd} X \arrow[r, hook, "\sim"] & X_{fib} & (X_{fib})_{cof} \arrow[l, two heads, "\sim"'] \end{tikzcd} \right)$$

Let us write M_{cof}, M_{fib} and M_{cf} for the full subcategories of M spanned by cofibrant, fibrant and bifibrant objects.

- * As we will see, the main point of model structures is to give tools to work with the localisation $M[W^{-1}]$. In particular, (co)fibrations are less important than weak equivalences and are a technical tool to access $M[W^{-1}]$.

- * By the lifting properties, the 3 classes of maps are determined by either:
  - W and Cof

- W and Fib
- Cof and Fib (Exercise)

So to "construct a model structure", one usually proceeds by choosing W and one of Cof or Fib, define Fib or Cof by lifting, and check the axioms. This can be quite difficult, and almost always relies on the small object argument.

Sometimes with additional work, one can get concrete descriptions of both Cof and Fib, or at least of Cof and all fibrant objects (for instance).

* In fact the structure is determined by even less:

Prop 44: (Joyal) Let M be a model category. The whole model structure is determined by the cofibrations and the fibrant objects (or by the fibrations and the cofibrant objects).

$\square$

* The factorisations in the WFS can usually be chosen to be functorial, because they are usually constructed with the small object argument. This makes a few proofs simpler but does not really affect the theory, and in does not make any difference in practice.

* The axioms of model categories were abstracted by Quillen from properties of the homotopy theory of three central examples:

- topological spaces
- simplicial sets
- chain complexes.

I will discuss very briefly all three.

Prop 45: In sSet, we have two weak factorisation systems:

- (Monomorphisms, Trivial fibrations)
- (Anodyne morphisms, Kan fibrations).

proof: We have already proved this as an application of the small object argument! See Corollary III.1.14.

↑

developed by (Grothendieck and) Quillen for this kind of purpose. □

Thm 46: There is a unique model structure on sSet, the Kan-Quillen model structure, such that:

- the cofibrations are the monomorphisms.
- the fibrations are the Kan fibrations.
- the weak equivalences are the maps $f: X \to Y$ such that that, for all K Kan complex,

$$f^* : \underset{\pi_0 \operatorname{Map}(Y,K)}{\underset{\shortparallel}{[Y, K]}} \xrightarrow[\text{bij.}]{\sim} \underset{\pi_0 \operatorname{Map}(X,K)}{\underset{\shortparallel}{[X, K]}} . \quad \left(\Longleftrightarrow |f| \text{ homotopy equivalence} \right)$$

- In the Kan-Quillen model structure, every object is cofibrant and the fibrant objects are the Kan complexes.

Sketch:

- The weak equivalences satisfy the 2-out-of-3 property: $X \xrightarrow{f} Y \xrightarrow{g} Z$

$$\begin{tikzcd}
{[Z,K]} \arrow[r, "g^*"'] \arrow[rr, bend left, "(gf)^*"] & {[Y,K]} \arrow[r, "f^*"'] & {[X,K]}
\end{tikzcd}$$

and apply 2-out-of-3 for bijections of sets.

- By the prevuous proposition, we have two WFS. To finish the proof, it suffices to show that:

I) A morphism is a trivial Kan fibration <u>iff</u> it is a Kan fibration and a weak equivalence.

II) A morphism is anodyne <u>iff</u> it is a monomorphism and a weak equivalence.

(One nice, purely simplicial proof of I) and II) relies on Kan's "Ex^∞ functor." which is an explicit fibrant replacement functor with good properties.

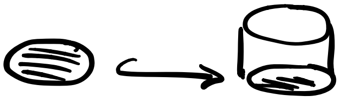
Here are references:

- I) : [Kerodon, Cor 3.3.7.4]
- II) : [Kerodon, Cor. 3.3.7.5]

□

Thm 47: There is a unique model structure on Top, the Quillen model structure, such that:

- the weak equivalences are ... the weak (homotopy) equivalences $!\left(\begin{matrix} \forall n \geq 0, \; f : X \to Y, \; x \in X, \\ \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y, f(x)) \end{matrix} \right)$
- the fibrations are the Serre Fibrations $\left(D^n \hookrightarrow D^n \times I \,\middle|\, n \geq 0 \right)^{\boxslash}$.



- the cofibrations are $\overline{\left(S^{n-1} \hookrightarrow D^n \,\middle|\, n \geq 0 \right)}$

(the "retracts of relative cell-complexes")

- In the Quillen model structure, every object is fibrant, and the cofibrant objects are the retracts of cell complexes.

sketch:

- Weak homotopy equivalences in Top satisfy the 2-out-of-3.
- The small object argument produces WFS:

$$- \left(\underset{\substack{\uparrow \\ \text{Cofibrations}}}{\overline{(S^{n-1} \hookrightarrow D^n)}}, (S^{n-1} \hookrightarrow D^n)^{\boxslash} \right)$$

$$- \left(\overline{(D^n \hookrightarrow D^n \times I)}, \underset{\substack{\uparrow \\ \text{fibrations}}}{(D^n \hookrightarrow D^n \times I)^{\boxslash}} \right)$$

- Again to finish the proof one must show compatibilities between these. The key case here is:

$$(D^n \hookrightarrow D^n \times I)^{\boxslash} \cap W = (S^{n-1} \hookrightarrow D^n)^{\boxslash}$$

which turns out to follow from standard arguments from the study of CW-complexes (what Peter May calls the HELP lemma: "homotopy extension and lifting properties"; see refs in the summary of the lecture).

$\square$

Rmk: There are other model structures on Top with the same weak equivalences; in particular if you prefer Hurewicz fibrations, you can have a look at [May-Ponto, §17.4].

Thm 48: Let R be a commutative ring.

(can do a lot more general abelian categories)

1) There is a unique model structure on $\mathrm{Cpl}_{R,\leq 0}$, the injective model structure, such that:

- the weak equivalences are the quasi-isomorphisms.
- the cofibrations are the levelwise monomorphisms. ($A_\bullet \to B_\bullet$ with $A_n \hookrightarrow B_n$)
- the fibrations are the levelwise epimorphisms with injective kernels ($A_\bullet \to B_\bullet$ with $0 \to \overset{\text{injective}}{I_n} \to A_n \to B_n \to 0$)

1) There is a unique model structure on $\mathrm{Cpl}_{R,\geq 0}$, the projective model structure, such that:

- the weak equivalences are the quasi-isomorphisms.
- the fibrations are the levelwise epimorphisms ($A_\bullet \to B_\bullet$ with $A_n \twoheadrightarrow B_n$)
- the cofibrations are the levelwise monomorphisms with projective cokernels.

$$\left(A_\bullet \longrightarrow B_\bullet \text{ with } 0 \to A_n \to B_n \to \overset{\text{projective}}{P_n} \to 0 \right)$$

$\square$

* There are also model structures for unbounded complexes but they are a little more subtle.

* These model structures on complexes are very useful to understand what is going on in homological algebra: fibrant replacements in the injective model structure are injective resolutions, etc.

- One first use of model categories for ∞-categories is to organise the comparison between different models.

Thm 49: (Joyal) There exists a unique model structure on sSet, the *Joyal model structure*, such that:

- the weak equivalences are the categorical equivalences.
- the cofibrations are the monomorphisms.
- the fibrant objects are the ∞-categories. $\square$

One can describe the structure in some more details, in particular the fibrations between fibrant objects (= *isofibrations*). We may come back to this - or not.

Thm 50: (Bergner) There exists a unique model structure on Cat_Δ, the **Bergner model structure**, such that:

- the weak equivalences are Dwyer-Kan equivalences.
- the fibrations are functors $F : C \to D$ such that
  * $\forall\, x, y \in C$, $Hom_C(x,y) \to Hom_D(F(x),F(y))$ is a Kan fibration.
  * $\forall\, x_1 \in C$, $y \in D$, and homotopy eq. $e : F(x_1) \to y$, there exists an homotopy eq. $d : x_1 \to x_2$ such that $F(d) = e$.

In particular, the fibrant objects are precisely the locally Kan simplicial categories.

$\square$

* Let's come back to the general theory.

What does having a model structure give you?

Quillen discovered that there is an intrinsic notion of homotopy between morphisms in a model category, which generalizes the notions in Top and sSet, and also chain homotopies in Cpl_R.

(In practice, this intrinsic notion is not essential because one uses enriched model categories, as we will see soon.)

* Nevertheless, Quillen defines, for

$$C \in M_{cof} \quad \text{and} \quad F \in M_{fib} :$$

$$[C, F] := M(C, F)/_{\text{homotopy}}.$$

and the homotopy category $\mathrm{Ho}(M)$:

- $\mathrm{Ob}\ \mathrm{Ho}(M) = \mathrm{Ob}(M_{cf})$
- $\mathrm{Ho}(M)(X, Y) = [X, Y]$

(He also proves that composition is well-defined up to homotopy, etc.)

* Recall also the notion of localisation of a category C at a class of maps W: it is the universal category $C[W^{-1}]$ equipped with a functor $C \longrightarrow C[W^{-1}]$ which sends morphisms in W to isomorphisms:

$$\begin{tikzcd}
W \arrow[ddr, mapsto] & C \arrow[rr] \arrow[dr] & & C[W^{-1}] \arrow[dl, dashed, "\exists!"] \\
& & D & \\
& \text{isos} & &
\end{tikzcd}$$

We can then state:

Thm 51: (Quillen's fundamental theorem)

M model category.

1) Let $C \in M_{cof}$, $F \in M_{fib}$.

Then $$[C, F] \xrightarrow{\sim} M[W^{-1}](C, F)$$

2) The inclusion $M_{cf} \hookrightarrow M$ induces an equivalence of categories $\mathrm{Ho}(M) \xrightarrow{\sim} M[W^{-1}]$.

$\square$

Since every object admits (co)fibrant replacements, this gives in principle a way to compute in $M[W^{-1}]$.

Ex 52

- For Top with the Quillen model structure, this is closely related with two well-known foundational facts in algebraic topology: CW-approximation and the Whitehead theorem.

(because CW - complexes $\subset \mathrm{Top}_{cof}$ = retracts of cell complexes)

We now discuss the functoriality of model categories.

Def : Let M, N be model categories.

- A Quillen adjunction is an adjunction

$$F : M \underset{\perp}{\rightleftarrows} N : G$$

such that $\begin{cases} F(Cof) \subset Cof \\ G(Fib) \subset Fib \end{cases}$

Lemma : $F \dashv G$ is a Quillen adjunction

$$\Updownarrow$$

$$F(Cof) \subset Cof \quad \text{and} \quad F(Cof \cap W) \subset Cof \cap W$$

$$\Updownarrow$$

$$G(Fib) \subset Fib \quad \text{and} \quad G(Fib \cap W) \subset Fib \cap W.$$

proof : Exercise.

(eg any small ∞-category can be embedded as a full subcategory of a presentable ∞-category by the ∞-version of the Yoneda lemma).